The trigonometric Rosen–Morse potential in the supersymmetric quantum mechanics and its exact solutions

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Abstract

The analytic solutions of the one-dimensional Schrödinger equation for the trigonometric Rosen–Morse potential reported in the literature rely upon the Jacobi polynomials with complex indices and complex arguments. We first draw attention to the fact that the complex Jacobi polynomials have non-trivial orthogonality properties which make them uncomfortable for physics applications. Instead we here solve the above equation in terms of real orthogonal polynomials. The new solutions are used in the construction of the quantum-mechanical superpotential.

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1. Introduction

Supersymmetric quantum mechanics was originally proposed by Witten [1] as a simple learning ground for the basic concepts of supersymmetric quantum field theories but soon after it evolved into a research field on its own rights. Supersymmetric quantum mechanics starts with the factorization of one-dimensional Hamiltonians,

$$H(y) = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V(y), \tag{1}$$

according to $H(y) = A^+(y)A^-(y) + \epsilon$ with $A^\pm(y) = \left(\pm\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}y} + U(y)\right)$ where $U(y)$ is the superpotential and ϵ being a constant. Next it proves that if $\psi_n(y)$ is an exact solution to the $H(y)$ eigenvalue problem, $H(y)\psi_n(y) = E_n\psi_n(y)$, then $A^-(y)\psi_n(y)$ is an eigenfunction to $\widetilde{H}(y) = A^-(y)A^+(y) + \epsilon$ corresponding to the same eigenvalue. In other words, knowing the superpotential allows one to generate the $\widetilde{H}(y)$ spectrum from the spectrum of $H(y)$ and vise versa. Moreover, in the case of zero ground state (gst) energy, knowing $U(y)$ allows one to

solve $A^-(y)\psi_{\text{gst}}(y) = 0$ and obtain the ground state wave function. Knowing the ground state wave function then allows one to recover the superpotential as

$$U(y) = -\frac{\hbar}{\sqrt{2m}} \frac{\mathrm{d}}{\mathrm{d}y} \ln \psi_{\text{gst}}(y). \tag{2}$$

At that stage one employs the isospectral pair of operators H, and $\widetilde{H}$ in the construction of the super-Hamiltonian $\mathbf{H}$ as $\mathbf{H} = \mathrm{diag}(H, \widetilde{H})$ and upon introducing 'supercharges' as

$$Q = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}, \tag{3}$$

proves them to satisfy the following superalgebra:

$$\{Q, Q^\dagger\} = \mathbf{H}, \qquad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \qquad [Q, \mathbf{H}] = [Q^\dagger, \mathbf{H}] = 0. \tag{4}$$

The ground state wave function to $\mathbf{H}$ is defined as

$$\Psi_{\text{gst}} = \begin{pmatrix} \psi_{\text{gst}} \\ \widetilde{\psi}_{\text{gst}} \end{pmatrix}. \tag{5}$$

The relationship to the field-theoretic SUSY is then established through the observation that in case $A^-\psi_{\text{gst}} \neq 0$ then the supercharges do not annihilate the respecive vacua, $Q\Psi_{\text{gst}} \neq 0$, and $Q^\dagger\widetilde{\Psi}_{\text{gst}} \neq 0$, and SUSY is spontaneously broken. On the contrary, when $A^-\psi_{\text{gst}}(y) = 0$, then the supercharges annihilate the ground state which is a signature for the absence of charge condensates there, and thereby for SUSY realization in the manifest (multiplet) Wigner–Weyl mode. In this manner supersymmetric quantum mechanics relates to SUSY in field theory where the róle of Q and $Q^\dagger$ is taken by boson-fermion (and vise versa) ladder operators.

The above considerations clearly reveal the importance of knowing the exact solutions of the quantum-mechanics Hamiltonians. These solutions are furthermore important in the construction of higher dimensional charge algebras with more but two Hamiltonians (hierarchy of Hamiltonians) [2]. The supersymmetric quantum mechanics manages a family of exactly soluble potentials [3], one of them being the trigonometric Rosen–Morse potential (tRMP). As long as this potential is obtained from the Eckart potential [4] by complexification of the argument and one of the constants, also its solutions have been concluded from those of the Eckart potential by the same procedure. In doing so, one ends up with Jacobi polynomials with complex indices and complex arguments. However, such complex polynomials are not comfortable for physical applications basically in view of their non-trivial orthogonality properties [5, 6].

We here make the case that the trigonometric Rosen–Morse potential is exactly soluble in terms of a family of *real* orthogonal polynomials and present the solutions.

The paper is organized as follows. In the next section we present the tRMP derivation from the Eckart potential and draw attention to the non-trivial orthogonality properties of the Jacobi polynomials with complex parameters and arguments. In section 3 we solve analytically the one-dimensional Schrödinger equation with the trigonometric Rosen–Morse potential and present the solutions. In section 4 we employ the exact ground state wave function in the construction of the tRMP superpotential. The paper closes with brief concluding remarks.

2. The trigonometric Rosen–Morse potential as a complexified Eckart potential

Before proceeding further we first introduce a properly chosen length scale d and change variables in the one–dimensional Schrödinger equation (1) to dimensionless ones according to

$$z = \frac{y}{d}, \qquad v(z) = V(dz)/(\hbar^2/2md^2), \qquad \epsilon_n = E_n/(\hbar^2/2md^2). \tag{6}$$
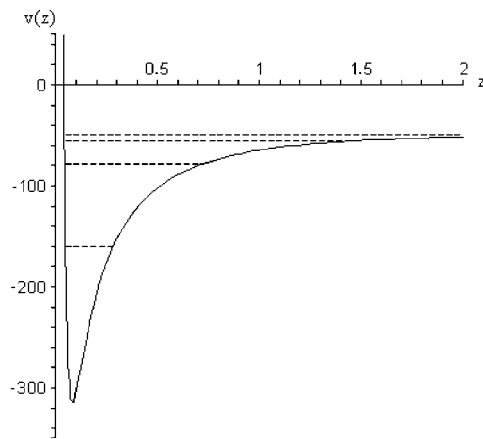
Figure 1. Eckart potential. The solid curve represents the potential while the dashed lines are the energy levels. Note that the argument of this potential is unbound from above, i.e. $0 < z < \infty$ and the number of bound states is limited. The parameters of the displayed potential take the values $a = -1, b = 50$.

Next we employ the Eckart potential [4],

$$v(z) = -2b \coth z + a(a - 1) \operatorname{csch}^2 z, \tag{7}$$

where $b > a^2$. The exact solutions to the Eckart potential are known and read:

$$\psi_n(x) = c_n(x - 1)^{(\beta_n - n - a)/2}(x + 1)^{-(\beta_n + n + a)/2} P_n^{\beta_n - n - a, -(\beta_n + n + a)}(x),$$

$$x = \coth z, \qquad \beta_n = \frac{b}{n + a}. \tag{8}$$

Here, $P_n^{(\beta_n - n - a, -(\beta_n + n + a))}(x)$ are the well-known Jacobi polynomials [7, 8] with $n \leqslant (b^{1/2} - a)$, and c_n is a normalization constant. Equation (8) equivalently rewrites to

$$\psi_n(x) = c_n(x^2 - 1)^{-(n+a)/2} e^{-\beta_n \operatorname{arccoth} x} P_n^{(\beta_n - n - a, -(\beta_n + n + a))}(x). \tag{9}$$

The corresponding energy spectrum is determined by

$$\epsilon_n = -(n + a)^2 - \frac{b^2}{(n + a)^2}. \tag{10}$$

Form and energy spectra of the Eckart potential are illustrated by figure 1. Let us now complexify the argument of the Eckart potential and one of its constants according to

$$z \to -\mathrm{i}z, \qquad \text{or, equivalently,} \qquad x \longrightarrow \mathrm{i}x, \qquad b \to \mathrm{i}b. \tag{11}$$

Substitution of equation (11) into (7) results in

$$v(z) = -2b \cot z + a(a - 1) \csc^2 z, \tag{12}$$

and thereby in the trigonometric Rosen–Morse potential [3] shown in figure 2.

In the literature [3] the solution of the Schrödinger equation with the trigonometric Rosen–Morse potential is concluded from equation (9) through complexification of b and x leading to

$$\psi_n(\mathrm{i}x) = c_n((\mathrm{i}x)^2 - 1)^{-(n+a)/2} e^{-\beta_n \operatorname{arccoth} \mathrm{i}x} P_n^{((\mathrm{i}\beta_n - n - a), -(\mathrm{i}\beta_n + n + a))}(\mathrm{i}x). \tag{13}$$
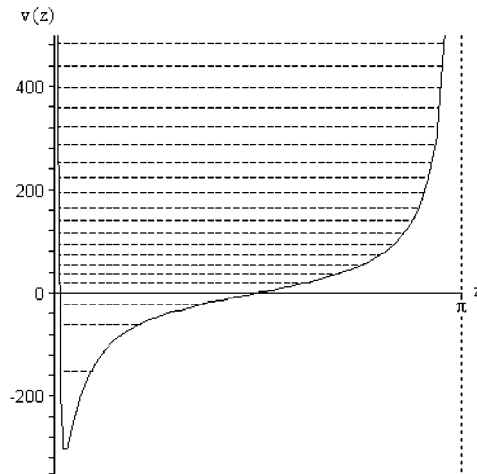
Figure 2. The trigonometric Rosen–Morse potential. The solid line represents the potential, while the dashed lines are the energy levels. Note that the argument of this potential is bound between $0 < z < \pi$ and the number of states is unlimited. The potential parameters take the values $a = 1, b = 50$.

In other words, these solutions need the Jacobi polynomials with complex indices and complex arguments. Unfortunately, the complex Jacobi polynomials are not very well suited for physics applications. Suffices to write down these polynomials explicitly,

$$P_n^{(A,B)}(\mathrm{i}x) = \frac{2^n (A+1)_n}{(n+A+B+1)_n} \quad {}_2F_1\left(-n, n+A+B+1; A+1 \Big| \frac{1-\mathrm{i}x}{2}\right),$$

$$
{}_2F_1\left(-n, n+A+B+1; A+1 \Big| \frac{1-\mathrm{i}x}{2}\right)
$$
$$
= \frac{\Gamma(A+1)}{\Gamma(n+A+B+1)\Gamma(-(n+B))} \int_0^1 t^{n+A+B}(1-t)^{-(n+B+1)} \left(1 - t\frac{1-\mathrm{i}x}{2}\right)^n \mathrm{d}t,
$$

$$A = \mathrm{i}\beta_n - n - a, \qquad B = -(\mathrm{i}\beta_n + n + a),$$

(14)

where ${}_2F_1(a, b; c|x)$ is the well-known hypergeometric function, and $(\cdots)_n$ is the Pochhammer symbol of the expression in the parentheses, in order to become aware of the calculational difficulties to be expected. It is obvious that one has to worry about the interplay between indices and integration contours, a subject studied in [5, 6]. There the authors claim dependence of the orthogonality properties on the indices and the integration contours. In order to avoid all those difficulties we here search for real solutions of the one-dimensional Schrödinger equation with the trigonometric Rosen–Morse potential.

3. The one-dimensional Schrödinger equation for the trigonometric Rosen–Morse potential

3.1. The Sturm–Liouville method and the Rodrigues formula

For the sake of self-sufficiency of the representation we here review in brief the basics of the Sturm–Liuoville technique for solving second order differential equations.

Table 1. Special functions and their characteristics.

Name	Symbol	$w(x)$	$s(x)$	Interval	Conditions
Hermite	$H_m(x)$	e^{-x^2}	1	$(-\infty, \infty)$	
Laguerre	$L_m^\nu(x)$	$x^\nu e^{-x}$	x	$[0, \infty)$	$(\nu > -1)$
Jacobi	$P_m^{(\nu,\mu)}(x)$	$(1-x)^\nu(1+x)^\mu$	$(1-x^2)$	$[-1, 1]$	$(\nu, \mu > -1)$
Gegenbauer	$C_m^\lambda(x)$	$(1-x^2)^{\lambda-1/2}$	$(1-x^2)$	$[-1, 1]$	$(\lambda > -1/2)$
Legendre	$P_m(x)$	1	$(1-x^2)$	$[-1, 1]$	
Chebyshev, type I	$T_m(x)$	$(1-x^2)^{-1/2}$	$(1-x^2)$	$[-1, 1]$	
Chebyshev, type II	$U_m(x)$	$(1-x^2)^{1/2}$	$(1-x^2)$	$[-1, 1]$	
This work	$C_{m+1}^{(a,b)}(x)$	$(1+x^2)^{-\mu} e^{-2\frac{b}{\mu}\cot^{-1} x}$	$(1+x^2)$	$(-\infty, \infty)$	$(\mu = m + 1 + a)$

The method of Sturm–Liouville [9] applies to differential equations of the type

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = -\lambda w(x)y, \tag{15}$$

and searches for the λ values that allow for a solution. The solution of equation (15), where $w(x)$ stands for the 'weight', or, 'density' function, are eigenfunctions of a Hermitian differential operator on the space of functions defined by the boundary conditions. Some special cases of the Sturm–Liouville type of differential equations allow for solutions by means of the Rodrigues formula [7]. To be specific, for $q(x) = 0$, and $p(x) = w(x)s(x)$, where $s(x)$ at most being a second order polynomial, the solution of equation (15) is given by a family of orthogonal polynomials. The classical polynomials of Hermite, Laguerre, Legendre and Jacobi are prominent examples for that.

In order to create the orthogonal polynomial solutions one considers the function [7]

$$C_m(x) = \frac{1}{w(x)}\frac{d^m}{dx^m}(w(x)s(x)^m), \tag{16}$$

with $C_1(x)$, $w(x)$ and $s(x)$ satisfying the following conditions:

1. $C_1(x)$ is a polynomial of first order,
2. $s(x)$ is a polynomial of at most second order and real roots,
3. $w(x)$ is real, positive and integrable within a given interval $[a, b]$, and satisfies the boundary conditions

$$w(a)s(a) = w(b)s(b) = 0. \tag{17}$$

The above three conditions seem quite restrictive indeed but nonetheless they allow for the construction of all the *classical* orthogonal polynomials reported in the standard textbooks [7–9]. The table shows all the ingredients of the Rodrigues formula required for the construction of the respective orthogonal polynomials.

If in addition one demands an orthonormalized set of polynomials, one has to introduce into equation (16) an additional constant, here denoted by K_m, according to

$$C_m(x) = \frac{1}{K_m w(x)}\frac{d^m}{dx^m}(w(x)s(x)^m). \tag{18}$$

In terms of $C_m(x)$, the Sturm–Liouville equation takes the form

$$\frac{d}{dx}\left(w(x)s(x)\frac{dC_m(x)}{dx}\right) = -\lambda_m w(x)C_m(x), \tag{19}$$

or, equivalently,

$$s(x)\frac{d^2 C_m(x)}{dx^2} + \frac{1}{w(x)}\left(\frac{ds(x)w(x)}{dx}\right)\frac{dC_m(x)}{dx} + \lambda_m C_m(x) = 0, \tag{20}$$

where

$$\lambda_m = -m\left(K_1\frac{dC_1(x)}{dx} + \frac{1}{2}(m-1)\frac{d^2 s(x)}{dx^2}\right). \tag{21}$$

Note that

$$C_1(x) = \frac{1}{K_1 w(x)}\left(\frac{ds(x)w(x)}{dx}\right). \tag{22}$$

All classical polynomials can be obtained through the above procedure and vise versa. The polynomials that can be obtained from the Rodrigues formula and which satisfy the three conditions mentioned above are necessarily the classical orthogonal polynomials, a result due to [10].

3.2. Solving Schrödinger's equation for the trigonometric Rosen–Morse potential

In this section we present the solution of the one-dimensional Schrödinger equation for the trigonometric Rosen–Morse potential as obtained in [11] and without any reliance on the complex Jacobi polynomials. For this purpose we first have to reshape Schrödinger's equation

$$\frac{d^2 R(z)}{dz^2} + (2b\cot z - a(a+1)\csc^2 z + \epsilon)R(z) = 0, \tag{23}$$

to the Sturm–Liouville form in equation (20). To do so we first change variables to

$$R(z) = e^{-\alpha z/2}F(z), \tag{24}$$

with α being a constant and then substitute in equation (23). After some simple algebraic manipulations one arrives at

$$\frac{d^2 F(z)}{dz^2} - \alpha\frac{dF(z)}{dz} + \left(2b\cot z - a(a+1)\csc^2 z + \left(\left(\frac{\alpha}{2}\right)^2 + \epsilon\right)\right)F(z) = 0. \tag{25}$$

Changing once more variables to $x = \cot z$ in which case $F(z)$ becomes a function of x denoted by $f(x)$, i.e.

$$F(z) \to f(x), \qquad x = \cot z, \tag{26}$$

equation (25) takes the form

$$\left(2bx - a(a+1)(1+x^2) + \left(\left(\frac{\alpha}{2}\right)^2 + \epsilon\right)\right)f(x) + (1+x^2)^2\frac{d^2 f(x)}{dx^2}$$
$$+ 2(1+x^2)\left(\frac{\alpha}{2} + x\right)\frac{df(x)}{dx} = 0. \tag{27}$$

Finally, upon substituting the factorization ansatz

$$f(x) = (1+x^2)^{-(1-\beta)/2}C(x), \tag{28}$$

into equation (27) and a subsequent division by $(1+x^2)^{(1-\beta)/2}$ one finds as intermediate result the equation

$$\left((-\beta(1-\beta) - a(a+1)) + \frac{-\alpha(1-\beta) + 2b)x + \left(\left(\frac{\alpha}{2}\right)^2 - (1-b)^2 + \epsilon\right)}{(1+x^2)}\right)C(x)$$
$$+ (1+x^2)\frac{d^2 C(x)}{dx^2} + 2\left(\frac{\alpha}{2} + \beta x\right)\frac{dC(x)}{dx} = 0. \tag{29}$$

If equation (29) is to coincide in form with equation (20), the following conditions have to be fulfilled:

$$-\alpha(1 - \beta) + 2b = 0, \tag{30}$$

$$\left(\frac{\alpha}{2}\right)^2 - (1 - \beta)^2 + \epsilon = 0. \tag{31}$$

Substitution of equations (30), (31) into equation (29) amounts to

$$(1 + x^2)\frac{d^2 C(x)}{dx^2} + 2\left(\frac{\alpha}{2} + \beta x\right)\frac{dC(x)}{dx} + (-\beta(1 - \beta) - a(a + 1))C(x) = 0. \tag{32}$$

Equation (32) relates to equations (20)–(22) via

$$-\beta(1 - \beta) - a(a + 1) = -m(2\beta + m - 1). \tag{33}$$

Now by determining β from equation (33), substituting it into equations (30), (31) and shifting m to $m \rightarrow n - 1$, the following n-dependent constants are found:

$$\beta_n = -(n + a) + 1, \qquad \alpha_n = \frac{2b}{n + a}, \tag{34}$$

$$\epsilon_n = (n + a)^2 - \frac{b^2}{(n + a)^2}, \tag{35}$$

with $n \geqslant 1$.

In this way one encounters $w(x)$ and $s(x)$ as

$$w_n(x) = (1 + x^2)^{-(n+a)} e^{-\alpha_n \operatorname{arccot} x}, \tag{36}$$

$$s(x) = 1 + x^2. \tag{37}$$

The polynomials which resolve the tRMP Schrödinger equation are now obtained in exploiting the Rodrigues formula (18) when rewritten in terms of n as

$$C_n^{(a,b)}(x) = \frac{1}{K_n w(x)} \frac{d^{n-1}}{dx^{n-1}}(w_n(x)s(x)^{n-1}). \tag{38}$$

The lowest $C_n^{(a,b)}(x)$ polynomials obtained in this fashion read

$$C_1^{(a,b)}(x) = \frac{1}{K_1}, \tag{39}$$

$$C_2^{(a,b)}(x) = \frac{2}{K_2}\left(-(1 + a)x + \frac{b}{2 + a}\right), \tag{40}$$

$$C_3^{(a,b)}(x) = \frac{2}{K_3}\left((1 + a)(2a + 3)x^2 - 2(2a + 3)\frac{b}{3 + a}x + \left(\frac{2b^2}{(3 + a)^2} - (1 + a)\right)\right), \tag{41}$$

$$C_4^{(a,b)}(x) = \frac{4}{K_4}\left(-(1 + a)(2a + 3)(2 + a)x^3 + 3(a + 2)(2a + 3)\frac{b}{(4 + a)}x^2\right.$$
$$\left. - 3(2 + a)\left(2\frac{b^2}{(4 + a)^2} - (1 + a)\right)x + \left(\frac{2b^3}{(4 + a)^3} - (3a + 4)\frac{b}{4 + a}\right)\right), \tag{42}$$

$$C_5^{(a,b)}(x) = \frac{4}{K_5}\left((1+a)(2a+3)(2+a)(2a+5)x^4 - 4(2a+3)(2+a)(2a+5)\frac{b}{(5+a)}x^3\right.$$

$$+ 6(2+a)(2a+5)\left(\frac{2b^2}{(5+a)^2} - (1+a)\right)x^2$$

$$- 4(2a+5)\left(\frac{2b^3}{(5+a)^3} - (3a+4)\frac{b}{5+a}\right)x$$

$$\left.+ \left(\frac{4b^4}{(5+a)^4} - \frac{4b^2}{(5+a)^2}(3a+5) + 3(2+a)(1+a)\right)\right), \tag{43}$$

where $x = \cot z$.

The above polynomials solve exactly equation (32) which can be immediately cross-checked by back substituting equations (43) into equation (32). Employing symbolic mathematical programs is quite useful in that regard.

Note that the solution was found under less rigid requirements but the ones listed immediately after equation (16) above. Indeed,

- the roots of our $s(x)$ function are imaginary (this is the only place where the complexification of the Eckart potential seems to have left footprints),
- equation (17) turned out to be a rule that allows to be broken.

In that sense, the orthogonal functions found here must belong to a different class of orthogonal functions.

Their orthogonality is obtained as

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{s(x)}(w_n(x))^{1/2}C_n^{(a,b)}(w_{n'}(x))^{1/2}C_{n'}^{(a,b)} = \delta_{nn'}. \tag{44}$$

Equation (44) shows convincingly that the new solutions have well-defined orthogonality properties on the real axes, which qualifies them as comfortable wave functions in quantum-mechanics applications.

The orthogonality condition for the wave functions $R_n(z)$ reads

$$\int_0^\pi \mathrm{d}z\, R_n(z)(R_{n'}(z))^* = \delta_{nn'}. \tag{45}$$

A version interesting for physical application (see the concluding section) is the one with $a = 0$,

$$v(z) = -2b\cot z, \tag{46}$$

in which case the normalization constant is calculated as

$$K_n = \left(\frac{(n!)^2 n^3 (1 - \mathrm{e}^{-2\pi b/n})}{4b(b^2 + n^4)}\right)^{1/2}. \tag{47}$$

The associated energy spectrum is given then by

$$\epsilon_n = n^2 - \frac{b^2}{n^2}. \tag{48}$$

Correspondingly, the wave functions for this case simplify and are found as

$$R_1(z) = \mathrm{e}^{-bz}\sin z\, C_1^{(0,b)}(\cot z),$$

$$R_2(z) = \mathrm{e}^{-bz/2}\sin^2 z\, C_2^{(0,b)}(\cot z),$$

$$\cdots \tag{49}$$

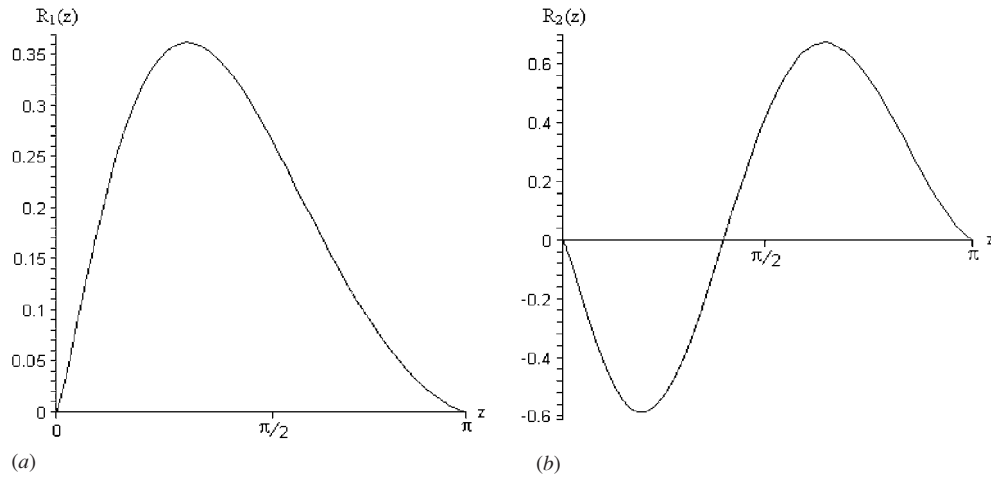$$R_n(z) = \mathrm{e}^{-bz/n}\sin^n z\, C_n^{(0,b)}(\cot z).$$

Figure 3. Wave functions for the first two levels in the trigonometric Rosen–Morse potential for $a = 0.25$ and $b = 1$.

Wave functions for the first two (unnormalized) levels with $a \neq 0$ are displayed in figure 3.

4. The trigonometric Rosen–Morse superpotential

In this section we derive the trigonometric Rosen–Morse superpotential from the exact ground state solution for which $a \neq 0$ has been calculated as

$$R_1(z) \propto \mathrm{e}^{-bz/(a+1)} \sin^{a+1} z. \tag{50}$$

Substitution of the latter equation (50) into equation (2) amounts to the following superpotential:

$$U(z) = -\frac{b}{a+1} + (a+1) \cot z, \tag{51}$$

shown in figure 4. Correspondingly, the $A^{\pm}(z)$ operators are obtained as

$$A^{\pm}(z) = \pm\frac{\mathrm{d}}{\mathrm{d}z} + (a+1)\cot z - \frac{b}{a+1}. \tag{52}$$

The corresponding Hamiltonian is then obtained identical to equation (12), as should be. The supersymmetric companion of $H(z)$, which is $\widetilde{H}(z)$, becomes

$$\widetilde{H}(z) = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 2b\cot z + (a+1)(a+2)\csc^2 z, \tag{53}$$

and one finds the supersymmetric companions to the solutions as

$$\widetilde{R}_n(z)|_{(a,b)} = R_{n-1}(z)|_{(a+1,b)}, \tag{54}$$

where $n > 1$, and $R_{n-1}(z)|_{(a+1,b)}$ are among the exact solutions of the trigonometric Rosen–Morse potential.

As a further possible application of the solutions found here we wish to mention the construction of the so-called hierarchy of Hamiltonians [2], where one needs to have at ones disposal exact orthonormalized functions for all levels because in this case one can pick up any energy level, ϵ_n, and its wave function $R_n(z)$. Obviously, equations (49) fully qualify for that purpose.
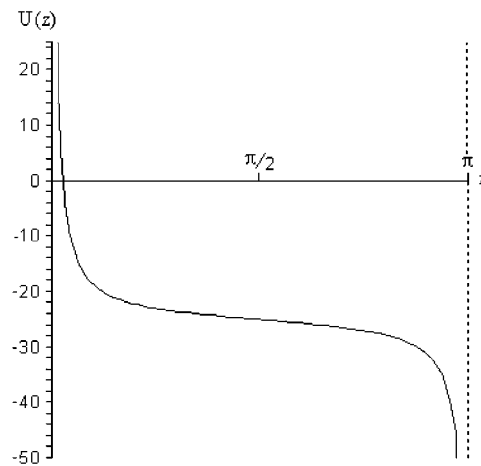
Figure 4. The trigonometric Rosen–Morse superpotential. The values of the displayed superpotential parameters are the same as in figure 2.
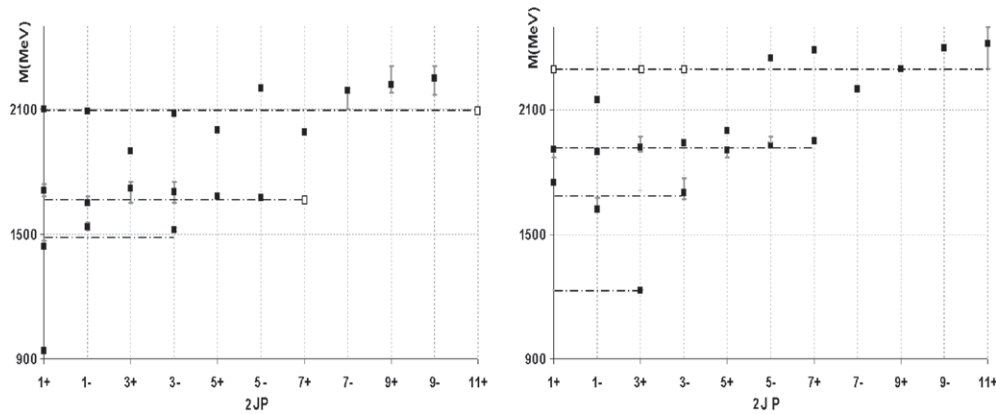

Figure 5. Experimentally observed baryon resonances (lhs) N and (rhs) Δ. The dash-point lines represent the mass average. Note that the resonances with masses above 2000 MeV are of significantly lower confidence but those with masses below 2000 MeV where the degeneracy is very well pronounced. Empty squares denote predicted ('missing') states.

5. Concluding remarks

To recapitulate, the knowledge on the real exact solutions of the one-dimensional Schrödinger equation for the trigonometric Rosen–Morse potential obtained in this work allows for a straightforward construction of all the necessary ingredients of the supersymmetric quantum mechanics. Compared to the textbook complex solutions, the exact real solutions reported here have the advantage of significantly simplifying the calculations.

In addition, the trigonometric Rosen–Morse potential is more but just one of the few exactly soluble quantum-mechanical potentials. It joins as a new member of the smaller and important group of potentials which generate physically relevant spectra. Indeed, in particle physics one encounters [12] the measured excitation spectra of the nucleon (N), and the so-called Δ particle, in turn displayed in figure 5.

According to an observation due to [13, 14] and references therein, those spectra repeat with an amazing accuracy the degeneracy patterns of the levels of the electron with spin in the hydrogen atom but are characterized by very different mass splittings. The following mass formula describes pretty well the averaged positions of the three narrow mass bands containing the series of $(n-1)$ parity doubled states (with $n=2, 4$ and 6) and with spins ranging from $\frac{1}{2}^{\pm}$ to $\left(n-\frac{3}{2}\right)^{\pm}$ together with the one unpaired maximal spin $\left(n-\frac{1}{2}\right)^{P}$ state of either natural, or, unnatural parity,

$$M_{(n;I)} - M_{(n;I)}^0 = g_I \frac{n^2-1}{4} - f_I \frac{1}{n^2}, \qquad I = N, \Delta. \tag{55}$$

Comparison of the baryon mass formula with equation (48) reveals coincidence between the baryonic and tRMP spectra. The underlying constituent dynamics of the excited baryons has been found to be that of a quark-di-quark system [13]. In fact, in order to describe the particle spectra one needs to solve the three-dimensional Schrödinger equation but this does not cause much problems because upon separation of the variables in polar coordinates the radial part of the three-dimensional Schrödinger equation reduces to that very same one-dimensional Schrödinger equation (23) up to the centrifugal barrier term, $l(l+1)/z^2$ (work in progress). All in all, the trigonometric Rosen–Morse potential and its real orthogonal polynomial solutions open new venues in the calculation of interesting observables in both supersymmetric quantum mechanics and particle spectroscopy.

Acknowledgments

References

[1] Witten E 1981 *Nucl. Phys.* B **188** 513
[2] Sukumar C V 1998 *J. Phys. A: Math. Gen.* **18** 2917
    Sukumar C V 2005 *Supersymmetries in Physics and Applications (AIP Proceedings* vol 744*)* ed R Bijker *et al* (New York: American Institute of Physics) p 167
[3] Cooper F, Khare A and Sukhatme U P 2001 *Supersymmetry in Quantum Mechanics* (Singapore: World Scientific)
[4] De R, Dutt R and Sukhatme U 1992 *J. Phys. A: Math. Gen.* **25** L843
[5] Kuijlaars A B J, Martinez-Finkelshtein A and Orive R 2003 Orthogonality of Jacobi polynomials with general parameters *Preprint* math.CA/0301037
[6] Beckermann B, Coussement J and Van Asshe W 2003 *Preprint* math.CA/0311055
[7] Dennery P and Krzywicki A 1996 *Mathematics for Physicists* (New York: Dover)
[8] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* 2nd edn (New York: Dover)
[9] http://en.wikipedia.org/wiki/Main_Page
[10] Cryer C W 1970 *Boll. Un. Mat. Ital.* **25** 1
    Álvarez Nodarse Renato *Las funciónes especiales y la fisica matematica* (in Spanish) http://euler.us.es/renato/clases.html
[11] Cliffor Benjamin Compean Jasso 2005 Baryon spectra in a quark–di-quark model with the trigonometric Rosen–Morse potential *MS Thesis* Institute of Physics, Autonomous University of San Luis Potosí, México (in Spanish)
[12] Alvarez-Gaumé L *et al* 2004 *Phys. Lett.* B **592** 1
[13] Kirchbach M, Moshinsky M and Smirnov Yu F 2001 *Phys. Rev.* D **64** 114005
[14] Kirchbach M 2004 *Rev. Mex. Fis.* **50** 54